Slogan Dieudonné theory : give a linear presentation of theory of Hopf algebras.

Hopf rings = graded ring objs in cat of Hopf algs.
$\simeq$ " " " in cat of Dieudonné modules

Compute Hopf rings for E_* = compute $E_*\Omega^2 S^3$.

Plan

1. What is a Hopf ring?
2. What is the Dieudonné modules? What are graded ring objs in this cat?
3. Why computing Hopf rings for E_* is equivalent to computing $E_*\Omega^2 S^3$?

References

[EP19] Formal Geometry & Bordism Operations by Eric Peterson (2019), Cambridge studies in advanced math, vol. 177.

[PG99] Hopf Rings, Dieudonné Modules, and $E_*\Omega^2 S^3$. by Paul G. Goerss (1999). Contemporary Math.

I. Hopf Rings.

Hopf alg : group object in the cat of coalgebras
Hopf ring : graded ring object in the cat of coalgebras.

cocomm. coassoc. assoc.

Explicitly. Hopf alg = assoc. & coassoc. bialgebra w/ antipode

i.e. (H, u, η, m, μ, S) over R

$$\begin{tikzcd} H\otimes H \arrow[rr, "S\otimes id"] \arrow[d, "m"'] & & H\otimes H \arrow[d, "m"] \\ H \arrow[r, "\eta"] \arrow[d, "\mu"'] & R \arrow[r, "u"] & H \arrow[d, "\mu"] \\ H\otimes H \arrow[rr, "id\otimes S"] & & H\otimes H \end{tikzcd}$$

u = unit, η = counit, m = multiplication μ = comultiplication.

S = k-linear antipode.

Why care about?

- If X Ω-spectrum or infinite loop space (which gives a Ω-spectrum)

  $X = \{X_i\}_{i\in\mathbb{N}}$. X_i has operation $*: X_i \times X_i \longrightarrow X_i$.

  Let E = generalized hom theory w/ Künneth iso, then "$*$" induces

  $$E_* X_i \otimes E_* X_i \longrightarrow E_*(X_i \times X_i) \longrightarrow E_* X_i$$

  together w/ diagonal map $\Delta: X_i \longrightarrow X_i \times X_i$ gives $E_* X_i$ the structure of Hopf alg.

- If X ring spectrum, $\forall$ space Y, $X^* Y = [Y, X_*]$, and

  $$\circ: X^k Y \times X^\ell Y \longrightarrow X^{k+\ell} Y$$

  corresponds to $\circ: X_k \times X_\ell \longrightarrow X_{k+\ell}$.

  Apply E^* = generalized coh thy w/ Künneth iso. $E^* X_i$ has a Hopf ring structure.

- $E_* BP$. $E_* MU$ for E cpx ori. coh. thy.

  $K(n)_* k(n)$

  $H_* KO$. $H_* BU$

  $H_*(QS^0; \mathbb{Z}/2)$ or $H_*(QS^0; \mathbb{Z}/p)$ Hopf alg over Dyer-Lashof alg.

II. Dieudonné theory & formal gps.

Formal gps arises naturally in the research of cpx ori. coh. thy, e.g. MU. Recall that a f.g.l. $F \in R[\![x, y]\!]$ is a power series satisfying

1) $F(x, y) = F(y, x)$

2) $F(x, 0) = F(0, x) = x$

3) $F(x, F(y, z)) = F(F(x, y), z)$

e.g. additive f.g.l. $F_a(x, y) = x + y$

multiplicative f.g.l. $F_m(x, y) = x + y + xy$.

<u>Recall</u> A formal gp $\hat{G}$ over R is an abelian gp obj in the cat of étale sheaves over Spec R, which is locally in Zariski top of the form $\hat{G}_F$ for F f.g.l., and

$$\hat{G}_F : \mathrm{CRing} \longrightarrow \mathrm{Ab}$$
$$A \longmapsto \mathrm{Nil}(A)$$

nilpotent elems of A w/ multiplication

$(x, y) \longmapsto F(x, y)$. $F \in A[\![x, y]\!]$.

$\hat{G}$ is a formal scheme. in fact the formal affine line $\hat{\mathbb{A}}^1_R = \mathrm{Spf}(R[\![x]\!])$.
i.e. the formal spectrum of $R[\![x]\!]$ w/ I-adic top.

Consider the algebraic de Rham complex $\Omega^*_{\hat{G}/A}$. of which terms are coherent sheaves over $\hat{G}$. Let $\omega \in \Omega^1_{\hat{G}/A}$ is called invariant differential if it invariant under right multiplication. That is,

$$m: \hat{G} \times \hat{G} \longrightarrow \hat{G} \quad \text{multiplication}$$

$$\pi_1: \hat{G} \times \hat{G} \longrightarrow \hat{G} \quad \text{projection}$$

one has $\pi_1^* \omega = m^* \omega$.

Thm Suppose $\hat{G} = \hat{G}_F$ for some f.g.l. $F \in R[\![x,y]\!]$.
ω inv. diff $\iff$ $\omega(x) = \omega(F(x,y)) \frac{\partial F}{\partial x}(x,y)$.

Thm $\forall$ formal gp $\hat{G}$ over $\mathbb{Q}$-alg A is locally iso to $\hat{G}_a$.
$\omega \in \Omega^1_{\hat{G}/A}$ gives rise to logarithm via integration. $\omega_a = dx$.

However, when A is not $\mathbb{Q}$-alg but of positive char p. this doesn't hold (actually leads to height p). Need some generalization of inv. diff.

IDEA $m: \hat{G} \times \hat{G} \longrightarrow \hat{G}$ multiplication

$\pi_i: \hat{G} \times \hat{G} \longrightarrow \hat{G}$ projection. $i = 1, 2$.

inv. diff. is the elet in $\ker(m^* - \pi_1^* - \pi_2^*)$.

Denote $PH^1_{dR}(\hat{G}/A) \subseteq H^1_{dR}(\hat{G}/A)$ be this kernel.

Thm (Poincaré lemma)

A p-local torsion-free ring. $f_1, f_2 \in xA[\![x]\!]$. If $f_1 \equiv f_2 \bmod p$ then $\forall\, w \in A[\![x]\!]dx$, $f_1^*(w) - f_2^*(w)$ is exact.

We're good to define the Dieudonné modules. Let $W_p(k)$ = p-typical Witt vectors on k perfect field. Then $W_p(k)$ is torsion-free. Recall in deformation theory. $W_p(k)$ is universal infinitesimal thickening of k. Explicitly. $\mathrm{Def}(A) \cong \mathrm{Hom}_k(W_p(k)[\![v_1, \dots, v_{n-1}]\!], A)$ (Lubin-Tate).

Quick Reminder: $W(A) \xrightarrow{w} A^{\mathbb{N}_0}$. $w(a) = (w_1(a), w_2(a), \dots)$

$w_n = a_0^{p^n} + p a_1^{p^{n-1}} + \cdots + p^n a_n$, $a = (a_0, a_1, \dots)$

Def k perfect, char $k = p > 0$. $\hat{G}_0$ = formal gp over k.
Lift $\hat{G}_0$ to $\hat{G}$ over $W_p(k)$. then the (contravariant) Dieudonné module of $\hat{G}_0$ is $D^*(\hat{G}_0) := PH^1_{dR}(\hat{G}/W_p(k))$.

Property: 1) Well-defined $\mathbb{Z}_p$-module.

2) Frobenius: $x \mapsto x^p \rightsquigarrow \hat{G}_0 \mapsto \mathrm{Frob}^*\hat{G}_0$

$\rightsquigarrow F: D^*(\hat{G}_0) \longrightarrow D^*(\mathrm{Frob}^*\hat{G}_0)$

Let φ lift of Frob on k to $W_p(k)$. Then

$$F(\alpha)(v) = \alpha^{\varphi} F(v)$$

3) Verschiebung : $V: \sum a_n x^{n-1} dx \longmapsto \sum a_{pn}^{\varphi^{-1}} x^{n-1} dx$

$$\Rightarrow \quad VF = p. \qquad \alpha V(v) = V(\alpha^{\varphi} v).$$

Thm D^* : cat of (smooth) formal gps over k of finite height p

$$\downarrow$$

cat of Dieudonné modules, i.e. modules over

$$\mathrm{Cart}_p = W_p(k) \langle F, V \rangle \Big/ \begin{pmatrix} FV = VF = p \\ Fw = w^{\varphi} F \\ wV = Vw^{\varphi} \end{pmatrix}.$$

is cat. equiv.

Prop Given a comm. gp scheme $\hat{G}$, one can recover the Hopf alg str on $k[\hat{G}]$.

Cor D^* : Hopf algs $\longrightarrow$ Dieudonné modules.

Ex Dieudonné theory can help in chromatic htpy thy.

Can use to compute Aut gp of Honda formal gp. Let Γ_d = Honda f.g.l. of ht d / k. $D^*(\Gamma_d) = \mathrm{Cart}_p / (\mathrm{Cart}_p \cdot (p - F^d))$, where the relation is given by $[p]_{\Gamma_d}(x) = x^{p^d}$.

$$\Rightarrow \quad \mathrm{End}\, \Gamma_d \cong W_p(\mathbb{F}_{p^d}) \langle F \rangle \Big/ \begin{pmatrix} Fw = w^{\varphi} F \\ F^d = p \end{pmatrix}$$

$$\Rightarrow \quad \mathrm{Aut}\, \Gamma_d \cong (\mathrm{End}\, \Gamma_d)^{\times}$$

This is important since one needs it to compute the E_d - based

Adams SS, which converges to $\pi_* \widehat{L_d \mathbb{S}}$ (d-localized sphere spectrum)

Rk. By Cartier duality, one can define the covariant Dieudonné theory D_*. The definition is basically the same, and is subject to

$$(D_* \hat{G}_{\Pi_0})^* \cong D^* \hat{G}_{\Pi_0}$$

F, V subject to the interchange rule.

III. More Usage.

<u>FACT</u> $D_*(-)$ is actually a graded module. The grading is given by D_* : Graded Hopf $\mathrm{Alg}^{f.\text{type}}_{\mathbb{F}_p} \longrightarrow$ Graded Mod

$$H \longmapsto \bigoplus_n D_n(H)$$

$$= \bigoplus_n \mathrm{Map}(H(n), H)$$

where $H(n) = \begin{cases} \text{free graded comm. Hopf alg} / \mathbb{F}_p \text{ w/ one gen. deg } n>0 \\ \mathbb{F}_p[x_0, x_1, \ldots, x_n] \text{ w/ Witt diagonal.} \end{cases}$

Else. $D_*(-)$ is rep. by p-typical Witt formal gp $\hat{W}_p$.

$\mathrm{Map}(D_* H(n), M) = M_n$.

In order to translate Hopf rings into Dieudonné modules, one needs a tensor product on Dieudonné modules:

$$D_*(M) \boxtimes D_*(N) \cong D_*(M \boxtimes N)$$

where for two Dieudonné modules A, B,

$$A \boxtimes B = \frac{\mathbb{Z}_p\langle F, V\rangle}{(VF = p)} \otimes_{\mathbb{Z}_p[V]} (A \otimes B) \Big/ \begin{pmatrix} 1 \otimes Fx \otimes y = F \otimes x \otimes Vy \\ 1 \otimes x \otimes Fy = F \otimes Vx \otimes y \end{pmatrix}$$

where $V(x \otimes y) = V(x) \otimes V(y)$.

- Now D_* : Hopf rings $\longrightarrow$ Dieudonné alg.

▲ Why care about it ?

<u>Thm</u> (Goerss - Lannes - Morel '93)

$X \to Y \to Z$ cofib seq of spectra. If $n > 1$,

$n \not\equiv \pm 1 \mod 2p$, then $\exists$ e.s.

$$D_n H_* \Omega^\infty X \longrightarrow D_n H_* \Omega^\infty Y \longrightarrow D_n H_* \Omega^\infty Z$$

Note that homology of infinite loop spaces is Hopf alg over Dyer - Lashof alg.

<u>Cor</u> $n > 1$, $n \not\equiv \pm 1$ mod $2p$. $\exists$ spectrum $B(n)$, n^{th} Brown - Gitler spectrum s.t. $B(n)_* X \xrightarrow{\cong} D_n H_* \Omega^\infty X$.

Properties of $B(n)$:

1) $B(n)$ connective & p-complete

2) $B = \{B(n)\}_{n \geq 0}$ graded comm. ring spectrum.

3) Let E = ring spectrum rep. E^* coh thy. Then

$$E_k B(n) \cong B(n)_k E \cong B(n)_n \Sigma^{n-k} E$$

$\rightsquigarrow$ $E_k B(n) \longrightarrow D_n H_* \Omega^\infty E(n-k)$

$\rightsquigarrow$ surjection $E_* B \xrightarrow{h} D_* H_* E_*$. at odd primes.

At $p=2$. h respects V & F.

4) $H^* B(n)$ is known for all p.

Let E be ring sp. $D_E \subseteq D_* H_* E_*$. $D_E = \{ D_{2m} H_* E(2n) \}$ be the Dieudonné ring. Suppose E_* torsion-free & concentrated in even degree.

Thm (Goerss '99)

Suppose $E_* B(n)$ is also concentrated in even degrees for all n. $e \in D_1 H_* E(1)$ be the image of the gen. of $\pi_1 E(1)$ under the Hurewicz map $E_* \to H_* E$. Then

$$D_E[e]/(e^2 - b_1) \xrightarrow{\cong} D_* H_* E$$

is an iso of Dieudonné algs.

If $E_* B(n)$ concentrated in even deg. $\forall n$. then

$$E_* B(ev) = \{ E_* B(2n) \} = \{ E_{2m} B(2n) \}$$

and $E_* B \to D_* H_* E$ restricts to iso of Dieudonné algs

$$E_* B(ev) \longrightarrow D_E.$$

- Relation to $E_* \Omega^2 S^3$.

$\Omega^2 S^3$ admits a filtration $\{ F_k \Omega^2 S^3 \}$.

Let $\Omega^2 S^3_+ = \Omega^2 S^3 \sqcup pt$. It admits a filtration $\{ F_k = F_k \Omega^2 S^3_+ \}$.

Snaith splitting: $\Omega^2 S^3_+ \simeq \bigvee_k F_k / F_{k-1}$

and $$F_k/F_{k-1} \simeq \begin{cases} \Sigma^{2n(p-1)} B(2n) & , \quad k = pn \\ \Sigma^{2n(p-1)+1} B(2n+1) & , \quad k = pn+1 \\ * & , \quad k \not\equiv 0, 1 \mod p \end{cases}$$

for $p > 2$

$$F_k/F_{k-1} \simeq \Sigma^k B(k) \quad . \qquad p = 2.$$

So the assoc. graded spectrum of $\Omega^2 S^3_+$ = regrading of $B = \{B(n)\}$.

Denote the regraded spectrum by B'. Then

$$E_* B(ev) \cong E_* B'.$$

Then use Adams SS to D_E through $E_* B'$. Goerss did $E = BP$ in his paper.